

Nonmatrix Varieties and Nil-Generated Algebras Whose Units Satisfy a Group Identity

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Let R^\times denote the group of units of an associative algebra R over an infinite field F . We prove that if R is unitarily generated by its nilpotent elements, then R^\times satisfies a group identity precisely when R satisfies a nonmatrix polynomial identity. As an application, we examine the group algebra FG of a torsion group G and the restricted enveloping algebra $u(L)$ of a p -nil restricted Lie algebra L . Giambruno, Sehgal, and Valenti recently proved that if the group of units $(FG)^\times$ satisfies a group identity, then FG satisfies a polynomial identity, thus confirming a conjecture of Brian Hartley. We show that, in fact, $(FG)^\times$ satisfies a group identity if and only if FG satisfies a nonmatrix polynomial identity. In the case of restricted enveloping algebras, we prove that $u(L)^\times$ satisfies a group identity if and only if $u(L)$ satisfies the Engel condition. © 1997 Academic Press

1. INTRODUCTION

Let R be an associative unitary algebra over a field F of characteristic $p \geq 0$. Recall that R is said to satisfy a polynomial identity whenever there exists a nontrivial element $f(x_1, \dots, x_m)$ of the free F -algebra generated

by $\{x_1, x_2, \dots\}$ such that $f(r_1, \dots, r_m) = 0$ for all $r_i \in R$, whereas the group of units R^\times of R is said to satisfy a group identity if there exists a nontrivial word $w(y_1, \dots, y_m)$ in the free group generated by $\{y_1, y_2, \dots\}$ such that $w(u_1, \dots, u_m) = 1$ for all $u_i \in R^\times$. There is ample evidence in the literature to suggest that there may be some general underlying relationship between group identities and polynomial identities. For example, Gupta and Levin [5] proved that R^\times is nilpotent whenever R is Lie nilpotent; Smirnov and Zalesskii [21, 23] established the fact that R^\times is soluble whenever R is Lie soluble (if $p \neq 2$); and Shalev [19] proved that R^\times satisfies the Engel condition whenever R does. In the other direction, it follows from a result of Valitskas [22] that a radical algebra over an infinite field is a PI-algebra whenever its adjoint group satisfies a group identity. In radical algebras the adjoint group R° of R with the group operation given by $x \circ y = x + y + xy$ plays the role of the unit group in a unitary algebra. Also along this vein, Giambruno, Sehgal, and Valenti [4] recently confirmed a conjecture of Brian Hartley by proving that the group algebra FG of a torsion group G over an infinite field F satisfies a polynomial identity whenever $(FG)^\times$ satisfies a group identity. Subsequently, Passman [12] gave necessary and sufficient conditions for $(FG)^\times$ to satisfy an identity (cf. Theorem 4.1).

Two questions thus seem immediately relevant:

1. Does R^\times satisfy a group identity whenever R satisfies a PI?
2. Does R satisfy a PI whenever R^\times satisfies a group identity?

However, it quickly becomes clear that these questions are posed too generally. Indeed, free algebras have only the trivial units corresponding to F^\times , and $M_2(F)$, the algebra of 2×2 matrices over F , satisfies the standard polynomial identity of degree 4 even though $\text{GL}_2(F)$ contains a nonabelian free group whenever F contains a transcendental element. Thus it is natural to concentrate on algebras containing many units and on polynomial identities not satisfied by matrices. A polynomial identity not satisfied by $M_2(F)$ is called a nonmatrix identity. It follows from well-known results that whenever R satisfies a nonmatrix identity, then R^\times satisfies a group identity. Namely, we have the following (we use brackets to denote Lie commutators and parentheses to denote group commutators):

PROPOSITION 1.1. *Let R be a unitary algebra over a field of characteristic $p \geq 0$. Suppose that R satisfies a nonmatrix identity. Then*

1. R^\times is soluble if $p = 0$ and
2. R^\times satisfies an identity of the form $(y_1, y_2)^{p^t} = 1$ if $p > 0$.

In any case, R^\times satisfies a group identity.

Proof. Suppose first that $p = 0$ and R satisfies a nonmatrix identity. Then by a theorem of Kemer [9] the ideal in R generated by the Lie commutators of the form $[[a, b], [c, d], e]$ is nilpotent. So, in particular, R is Lie soluble, and hence by Smirnov and Zalesskii's theorem mentioned above, we find that R^\times is soluble.

Now suppose that $p > 0$ and R satisfies a nonmatrix identity f , say. Let A be the relatively free algebra of rank 3 in the variety satisfying f . Then A is a finitely generated PI-algebra, so that by a theorem of Razmyslov and Braun [2] the Jacobson radical $J(A)$ of A is nilpotent. Also, $A/J(A)$ is a semiprimitive PI-algebra satisfying a polynomial identity not satisfied by $M_2(F)$; consequently, $A/J(A)$ is commutative. Therefore, A satisfies a polynomial identity of the form $([x_1, x_2]x_3)^{p^t} = 0$ for a suitable t . It follows that R also satisfies $([x_1, x_2]x_3)^{p^t} = 0$, and so R^\times satisfies

$$\begin{aligned}(y_1, y_2)^{p^t} - 1 &= ((y_1, y_2) - 1)^{p^t} = (y_1 y_2 y_1^{-1} y_2^{-1} - 1)^{p^t} \\ &= ([y_1, y_2] y_1^{-1} y_2^{-1})^{p^t} = 0.\end{aligned}$$

The primary goal of this paper is to demonstrate that the converse to Proposition 1.1 holds for the class of nil-generated algebras over an infinite field. Let $\mathcal{N}(R)$ denote the set of nilpotent elements in R . We say that a unitary algebra R is nil-generated if it is generated by $\{1\} \cup \mathcal{N}(R)$. For example, $M_n(F)$ is nil-generated, as is the group algebra of a group generated by p -elements.

THEOREM 1.2. *Let R be a nil-generated unitary algebra over an infinite field of characteristic $p \geq 0$. If R^\times satisfies a group identity, then $\mathcal{N}(R)$ forms a locally nilpotent ideal and R satisfies a nonmatrix identity.*

As a consequence of Proposition 1.1 and Theorem 1.2, we obtain the following characterizations:

THEOREM 1.3. *Let R be a nil-generated unitary algebra over a field of characteristic 0. Then the following statements are equivalent:*

1. R^\times satisfies a group identity;
2. R satisfies a nonmatrix identity;
3. R is Lie soluble; and
4. R^\times is a soluble group.

THEOREM 1.4. *Let R be a nil-generated unitary algebra over an infinite field of characteristic $p > 0$. Then the following statements are equivalent:*

1. R^\times satisfies a group identity;
2. R satisfies a nonmatrix identity;
3. R satisfies the polynomial identity $([x_1, x_2]x_3)^{p^t} = 0$ for some t ; and

4. R^\times satisfies the group identity $(y_1, y_2)^{p^t} = 1$ for some t .

The following corollary is an analogue of the classical theorem of Kaplansky [8], which states that every nil-algebra satisfying a PI is locally nilpotent.

COROLLARY 1.5. *Let R be a nil-algebra over an infinite field. The adjoint group R° of R satisfies a group identity if and only if R satisfies a nonmatrix identity. In this case R is locally nilpotent.*

Proof. It is well known that R can be embedded into a unitary F -algebra R_1 in such a way that $\{1\} \cup R$ generates R_1 and $R_1^\times \cong F^\times \times R^\circ$. Theorems 1.2–1.4 now imply the result. ■

As a further application of Theorem 1.4, we are able to study group algebras FG , where G is any torsion group, and restricted enveloping algebras $u(L)$, where L is any p -nil restricted Lie algebra. In particular, we deduce that $(FG)^\times$ satisfies a group identity if and only if FG satisfies a nonmatrix identity, whereas $u(L)^\times$ satisfies a group identity if and only if $u(L)$ satisfies the Engel condition. These results are detailed in Sections 4 and 5.

2. EXISTENCE OF A POLYNOMIAL IDENTITY

The following result, Corollary 2.2 of [4] (cf. Proposition 1 of [3]), plays a crucial role in the proof of our Theorem 1.2.

LEMMA 2.1. *Let R be a semiprime algebra over an infinite commutative domain, such that its group of units R^\times satisfies a group identity. Then for every nilpotent element $a \in R$, $bc = 0 \Rightarrow bac = 0$.*

Let $\mathcal{L}(R)$ denote the Levitzki radical of the algebra R ; that is, the unique maximal locally nilpotent ideal in R . Then $R/\mathcal{L}(R)$ is semiprime (see Section 10 of [7], for example) and $\mathcal{L}(R)$ is contained in $\mathcal{N}(R)$, the set of all nilpotent elements in R .

LEMMA 2.2. *Let R be a nil-generated unitary algebra over an infinite field. If R^\times satisfies a group identity, then $\mathcal{L}(R) = \mathcal{N}(R)$. Consequently, $R = F \cdot 1 + \mathcal{N}(R)$ and every finite subset of $\mathcal{N}(R)$ generates nilpotent subalgebra in R .*

Proof. Let $\bar{R} = R/\mathcal{L}(R)$. Because $\mathcal{L}(R)$ is contained in the Jacobson radical $J(R)$ of R , \bar{R}^\times is a homomorphic image of R^\times and, hence, satisfies the same group identity. We claim that $\mathcal{N}(\bar{R}) = 0$.

Let $\bar{b}, \bar{c} \in \bar{R}$ be such that $\bar{b}\bar{c} = 0$. By induction, it follows that whenever $\bar{r}_1, \dots, \bar{r}_k \in \bar{R}$, then $\bar{b}\bar{r}_1 \cdots \bar{r}_k\bar{c} = 0$. Indeed, we know that $(\bar{b}\bar{r}_1 \cdots \bar{r}_{k-1})\bar{c} = 0$ and hence Lemma 2.1 yields the result. Since \bar{R} is nil-generated, it follows that $\bar{b}\bar{R}\bar{c} = 0$.

To show \bar{R} has no nontrivial nilpotent elements, it suffices to show that the only square-zero element in \bar{R} is 0. Suppose then there exists $\bar{x} \in \bar{R}$ with $\bar{x}^2 = 0$. The above argument yields $\bar{x}\bar{R}\bar{x} = 0$, so that $\bar{R}\bar{x}\bar{R}$ is a nilpotent ideal. Hence $\bar{x} = 0$ and it follows that $\mathcal{N}(\bar{R}) = 0$, as claimed. This implies $\mathcal{L}(R) = \mathcal{N}(R)$. ■

PROPOSITION 2.3. *Let R be a nil-generated algebra over an infinite field. If R^\times satisfies a group identity, then R satisfies polynomial identity.*

Proof. Let $w(y_1, \dots, y_n) = 1$ be a group identity for R^\times . Put $k = F[t]$ and let A denote the completion of the free associative algebra $k\{x_1, \dots, x_n\}$. The free group $F_n = \langle y_1, \dots, y_n \rangle$ embeds into A by the well-known Magnus argument via the map φ induced by $\varphi(y_i) = 1 + tx_i$ (cf. [10, Section 5.5]). It follows easily that

$$1 \neq \varphi(w(y_1, \dots, y_n)) = 1 + \sum_{m=1}^{\infty} t^m p_m(x_1, \dots, x_n),$$

where each $p_m(x_1, \dots, x_n)$ is a homogeneous element of degree m in the free algebra $F\{x_1, \dots, x_n\}$. Not all these elements can be trivial, so assume $p_{m_0}(x_1, \dots, x_n) \neq 0$ in $F\{x_1, \dots, x_n\}$.

Now consider $S = \langle r_1, r_2, \dots, r_n \rangle$, the subalgebra in R generated by arbitrary elements r_1, \dots, r_n in $\mathcal{N}(R)$. Then S is nilpotent by Lemma 2.2, and so for each $\lambda \in F$ the map $t \rightarrow \lambda$, $x_i \rightarrow r_i$ induces a well-defined epimorphism ψ from the augmentation ideal of A to S . Since $w(y_1, \dots, y_n) = 1$ is a group identity for R^\times , upon application of ψ we obtain

$$1 = w(1 + \lambda r_1, 1 + \lambda r_2, \dots, 1 + \lambda r_n) = 1 + \sum_{m=1}^{\infty} \lambda^m p_m(r_1, \dots, r_n)$$

for each $\lambda \in F$. (The sum is finite.) Now, using the fact that F is infinite, a routine Vandermonde matrix argument implies that each $p_m(r_1, \dots, r_n) = 0$. Thus $p_{m_0}(x_1, \dots, x_n) = 0$ is a polynomial identity for S , and hence for all of $\mathcal{N}(R)$. However, $R = F \cdot 1 + \mathcal{N}(R)$, so that $p_{m_0}([x_1, x_2], \dots, [x_{2n-1}, x_{2n}])$ is a nontrivial polynomial identity for R . ■

3. EXISTENCE OF A NONMATRIX IDENTITY

To complete the proof of Theorem 1.2, it remains to prove the following:

PROPOSITION 3.1. *Let R be a nil-generated algebra over an infinite field F . If R^\times satisfies a group identity $\omega(y_1, \dots, y_n) = 1$, then R satisfies a nonmatrix identity.*

Proof. Recall that R is a PI-algebra by Proposition 2.3. To establish the statement about the existence of a nonmatrix identity is more involved and we shall require some reductions (cf. proof of Proposition 1 in [3]).

First let us point out that it is enough to show that $\mathcal{N}(R)$ satisfies a nonmatrix identity as R is a commutative extension of $\mathcal{N}(R)$ by Lemma 2.2.

Next, using the fact that the derived subgroup of a free group of rank 2 is free of countably infinite rank, we may also assume that ω is a word in two variables only. Furthermore, the substitution $y_1 = y_1 y_2$ and $y_2 = y_2 y_1$ allows us to assume that R^\times satisfies a group identity of the form

$$\omega(y_1, y_2) = (y_1 y_2)^{\alpha_1} (y_2 y_1)^{\beta_1} \cdots (y_1 y_2)^{\alpha_j} (y_2 y_1)^{\beta_j} (y_1 y_2)^{\alpha_{j+1}} = 1,$$

where $j \geq 1$ and the integers α_i and β_i are nonzero with the possible exception of α_{j+1} . The Magnus representation of ω now becomes

$$\omega(1 + x_1, 1 + x_2) = 1 + \sum_{m=1}^{\infty} p_m(x_1, x_2).$$

As in the proof of Proposition 2.5, it follows that $\mathcal{N}(R)$ satisfies each of the polynomial identities $p_m(x_1, x_2) = 0$ (some of which may be trivial). In order to prove the proposition, it suffices for us to show that at least one of the p_m is not also satisfied by $M_2(F)$. Let us suppose then to the contrary. It follows from the Magnus representation of ω that $\omega(1 + a, 1 + b) = 1$ for each choice of nilpotent a, b in $M_2(F)$. Notice as well that $a^2 = 0$ implies that $(1 + a)^n = 1 + na$, for each integer n . It is easy to see that the reduced form of ω is of the type

$$\omega(y_1, y_2) = y_1^{\gamma_1} y_2^{\delta_1} \cdots y_1^{\gamma_k} y_2^{\delta_k} y_1^{\gamma_{k+1}}$$

where $k \geq 1$ and the integers γ_i and δ_i are one of 1, -1 , 2, or -2 with the possible exception of γ_{k+1} , which is one of 1, -1 , or 0. Now fix two square-zero elements $a, b \in M_2(F)$. Then for every $\lambda \in F$ we have

$$\begin{aligned} \omega(1 + \lambda a, 1 + \lambda b) &= (1 + \gamma_1 \lambda a)(1 + \delta_1 \lambda b) \cdots \\ &\quad \cdots (1 + \gamma_k \lambda a)(1 + \delta_k \lambda b)(1 + \gamma_{k+1} \lambda a). \end{aligned}$$

However, we also have

$$\omega(1 + \lambda a, 1 + \lambda b) = 1 + \sum_{m=1}^l \lambda^m p_m(a, b),$$

where $l = 2k + 1$ unless $\gamma_{k+1} = 0$, in which case $l = 2k$. Comparing coefficients of λ^l we find that

$$(\gamma_1 \gamma_2 \cdots \gamma_k)(\delta_1 \delta_2 \cdots \delta_k)(ab)^{k+1} = 0.$$

In the case of characteristic $p \neq 2$, this yields $(ab)^{k+1} = 0$, which leads to the desired contradiction; for example, set $a = e_{12}$ and $b = e_{21}$.

It remains to consider the case of characteristic $p = 2$. Making the substitution $y_1 = y_1 y_2$ and $y_2 = y_1 y_3$ into the reduced form of ω above allows us to assume that R^\times satisfies

$$\omega_2(y_1, y_2, y_3) = (y_1 y_2)^{\gamma_1} (y_1 y_3)^{\delta_1} \cdots (y_1 y_2)^{\gamma_k} (y_1 y_3)^{\delta_k} (y_1 y_2)^{\gamma_{k+1}}.$$

Let us represent ω_2 by

$$\omega_2(1 + x_1, 1 + x_2, 1 + x_3) = 1 + \sum_{m=1}^{\infty} q_m(x_1, x_2, x_3).$$

As argued above, it suffices to show that $M_2(F)$ does not satisfy some q_m . Therefore, let us suppose otherwise and fix two square-zero elements $a, b \in M_2(F)$. Using the fact that the characteristic is 2, it is easy to check that $a + ba + ab + bab$ also has square zero. Evaluating the Magnus representation tells us

$$\omega_2(1 + \lambda a, 1 + \lambda b, 1 + \lambda(a + ba + ab + bab)) = 1.$$

Notice that in the reduced form of $\omega_2(y_1, y_2, y_3)$ the variables appear with exponents 1 or -1 only. Then because $(1 + a)^{-1} = 1 - a = 1 + a$, etc., it follows that $\omega_2(1 + \lambda a, 1 + \lambda b, 1 + \lambda(a + ba + ab + bab))$ is merely an ordered product of the terms $1 + \lambda a$, $1 + \lambda b$, and $1 + \lambda(a + ba + ab + bab)$ in which no two consecutive terms are equal. The triviality of the coefficient of the highest power of λ appearing in the resulting expansion leads to the fact that $(ab)^t = 0$ for some suitable $t > 0$. This in turn gives the desired contradiction. ■

4. GROUP ALGEBRAS

Let us now consider group algebras FG of a torsion group over an infinite field F of prime characteristic. In [4] it was shown that whenever $(FG)^\times$ satisfies a group identity, then FG satisfies a polynomial identity. Passman [12] subsequently characterized all torsion groups G such that $(FG)^\times$ satisfies a group identity. We are able to extend these results as follows:

THEOREM 4.1. *Let FG be a group algebra of a torsion group over an infinite field F of characteristic $p > 0$. Then the following are equivalent:*

1. $(FG)^\times$ satisfies a group identity;
2. FG satisfies a nonmatrix identity;

3. G contains a normal subgroup A such that G/A and (A, A) are finite and (G, G) is a p -group of finite exponent;
4. $[FG, FG]FG$ is nil of bounded index; and
5. $((FG)^\times, (FG)^\times)$ is a p -group of finite exponent.

Proof. Assume that (1) holds; we shall deduce (2). Let X be the set of p -elements in G and write P for the subgroup of G generated by X . Then the group algebra FP is nil-generated and $(FP)^\times$ satisfies a group identity. From Lemma 2.2 it follows that $\mathcal{N}(FP)$ is a locally nilpotent maximal ideal in FP . Therefore $\mathcal{N}(FP)$ coincides with the augmentation ideal of FP and P is a locally finite p -group. Using the normality of P in G , it follows that the ideal $\mathcal{N}(FP)FG$ in FG is also locally nilpotent. Now, according to Corollary 1.5, $\mathcal{N}(FP)FG$ satisfies a nonmatrix identity as the adjoint group of $\mathcal{N}(FP)FG$ satisfies the identities of $(FG)^\times$. Also, because the kernel of the canonical projection $FG \rightarrow F(G/P)$ is $\mathcal{N}(FP)FG$, it follows that $(F(G/P))^\times$ satisfies a group identity. Since G/P is a p' -group, it follows that G/P is abelian as is shown in the semiprime case of [4]. Now FG satisfies some nonmatrix identity by the fact that it is a commutative extension of an algebra satisfying a nonmatrix identity.

The equivalence (1) \Leftrightarrow (2) now follows from Proposition 1.1 (1) \Rightarrow (3) is the statement of Lemma 2.4 in [12]. The implication (3) \Rightarrow (4) follows as in the proof of Lemma 3.3 in [12]. The remaining implications (4) \Rightarrow (5) and (5) \Rightarrow (1), are clear. ■

5. RESTRICTED ENVELOPING ALGEBRAS

Let $u(L)$ be the restricted enveloping algebra of a restricted Lie algebra L over a field F of characteristic $p > 0$. Restricted enveloping algebras satisfying a polynomial identity were characterized independently by Passman [11] and Petrogradski [13]; see Lemma 5.3 below. A restricted Lie algebra L is said to be p -nil if for every $x \in L$ there exists a natural number n such that $x^{p^n} = 0$. We are interested here in characterizing p -nil restricted Lie algebras L for which $u(L)^\times$ satisfies a group identity. It follows from Jacobson's restricted analogue of the Poincaré–Birkhoff–Witt theorem (see [6]) that for such an L , $u(L)$ is nil-generated. Therefore, according to Theorem 1.4, $u(L)$ satisfies a nonmatrix identity precisely when $u(L)^\times$ satisfies a group identity. More specifically, we have:

THEOREM 5.1. *If L is a p -nil restricted Lie algebra over an infinite field of characteristic $p > 0$, then the following statements are equivalent:*

1. $u(L)^\times$ satisfies a group identity;
2. $u(L)$ satisfies a nonmatrix identity;

3. $[L, L]$ is bounded p -nil and L contains a restricted ideal A such that L/A and $[A, A]$ are finite dimensional;
4. $u(L)$ satisfies the Engel condition; and
5. $u(L)^\times$ satisfies an identity of the form $(y_1^{p^t}, y_2) = 1$ for some t .

In fact, in Theorem 5.1 we need only assume that L can be generated by p -nil elements.

COROLLARY 5.2. *Let L be a virtually (p -nil) restricted Lie algebra over an infinite field. If $u(L)^\times$ satisfies a group identity, then $u(L)$ satisfies a polynomial identity.*

Observe that some precondition on L is required in Corollary 5.2; indeed, the restricted enveloping algebra of a free restricted Lie algebra is a free associative algebra, and so has only the trivial unit group F^\times .

To prove Theorem 5.1, we shall make use of the result of Passman and Petrogradski mentioned above:

LEMMA 5.3. *Let L be a restricted Lie algebra. Then its restricted enveloping algebra $u(L)$ satisfies a polynomial identity if and only if L possesses a restricted ideal (or subalgebra) A such that*

1. A has finite codimension in L and
2. $[A, A]$ is finite dimensional and p -nil.

We shall also require Theorem 1.2 of [16].

LEMMA 5.4. *Let L be a restricted Lie algebra. Then its restricted enveloping algebra $u(L)$ satisfies the Engel condition if and only if*

1. L is nilpotent,
2. $[L, L]$ is bounded p -nil, and
3. L possesses a restricted ideal A such that L/A and $[A, A]$ are finite dimensional.

Only implications (2) \Rightarrow (3) and (3) \Rightarrow (4) in Theorem 5.1 do not follow directly from Theorem 1.4. To prove (3) \Rightarrow (4), assume that (3) holds. Observe that the centralizer C of $[A, A]$ in A is of finite codimension in A and is hence of finite codimension in L . Thus we may replace A by C , to assume that A is nilpotent of class 2. Now L is nilpotent-by-(finite dimensional and p -nil). It follows from a result of Shalev [20, Proposition 5.1] that L is nilpotent. Now Lemma 5.4 yields the fact that $u(L)$ satisfies the Engel condition. It remains then to prove the following lemma.

LEMMA 5.5. *If L is a p -nil restricted Lie algebra such that $u(L)$ satisfies a nonmatrix identity, then $[L, L]$ is bounded p -nil and L contains a restricted ideal A such that L/A and $[A, A]$ are finite dimensional.*

Proof. The existence of A follows immediately from Lemma 5.3. From Theorem 1.4, there exists some t such that $u(L)$ satisfies an identity of the form $([x_1, x_2]x_3)^{p^t} = 0$. As argued above, L must be nilpotent. It remains to prove that any linear combination of commutators in L is p -nil of bounded index.

Claim 5.6. For a sufficiently large integer k , L satisfies the identity

$$(x + y)^{p^k} = x^{p^k} + y^{p^k}.$$

Proof. Let c be the nilpotency class of L , and choose k large enough that $p^k \geq p^t c$. Consider $\lambda \in F$ and expand $(x + \lambda y)^{p^k}$ to get

$$(x + \lambda y)^{p^k} - x^{p^k} - \lambda^{p^k} y^{p^k} = \sum_{i \geq 1} \lambda^i h_i(x, y),$$

where each $h_i(x, y)$ in L is homogeneous in x, y of total degree p^k . Each h_i is a sum of elements of the form

$$[r_1^{p^{\alpha_1}}, \dots, r_l^{p^{\alpha_l}}]^{p^\alpha},$$

where $p^\alpha(\sum_j p^{\alpha_j}) = p^k$ and $r_j \in \{x, y\}$. If $\alpha \geq t$, then this restricted Lie monomial is zero. On the other hand, if $\alpha < t$, then

$$\sum_j p^{\alpha_j} = p^{k-\alpha} \geq p^{k-t+1} \geq pc \geq c + 1.$$

Therefore

$$\begin{aligned} [r_1^{p^{\alpha_1}}, \dots, r_l^{p^{\alpha_l}}] &= \left[r_1^{p^{\alpha_1}}, \underbrace{r_2, \dots, r_2}_{p^{\alpha_2}}, \dots, \underbrace{r_l, \dots, r_l}_{p^{\alpha_l}} \right] \\ &= \left[-r_2, \underbrace{r_1, \dots, r_1}_{p^{\alpha_1}}, \underbrace{r_2, \dots, r_2}_{p^{\alpha_2}-1}, \dots, \underbrace{r_l, \dots, r_l}_{p^{\alpha_l}} \right] \\ &= 0, \end{aligned}$$

being a commutator of length greater than c . ■

To finish the proof of Lemma 5.5, let r_i, s_i be arbitrary elements in L . Taking $p^k \geq p^t c$ as in the claim we have

$$\left(\sum_i \beta_i [r_i, s_i] \right)^{p^k} = \sum_i \beta_i^{p^k} [r_i, s_i]^{p^k} = 0,$$

as required. ■

Corollary 5.2 follows by combining Theorem 5.1 with Lemma 5.3.

6. CONCLUDING REMARKS

Let us close by observing that not every nil PI-algebra satisfies a nonmatrix identity. In light of our results, this is equivalent to the fact that the adjoint group of a nil PI-algebra need not satisfy a group identity.

PROPOSITION 6.1. *Let F be an infinite field of characteristic $p \geq 0$. Then there exists a locally nilpotent associative algebra R over F such that R satisfies a polynomial identity, and yet its adjoint group R° does not satisfy any group identity.*

Proof. For the case of $p > 0$, consider the restricted Lie algebra L generated by the set $\{x, y_1, y_2, \dots, z_1, z_2, \dots\}$, subject to the relations $[x, y_i] = z_i$ is central, $[y_i, y_j] = 0$, and $x^p = y_i^p = z_i^{p^i} = 0$, for all $i, j \geq 1$. Then the ideal of L generated by $\{y_1, y_2, \dots\}$ is abelian and of codimension 1 in L . Hence, $R = L(u(L))$ satisfies a polynomial identity by Lemma 5.3. Furthermore, L is locally-(finite dimensional and p -nil), so that R is locally nilpotent (see Lemma 2.4 of [16], for example). However, R° does not satisfy any group identity, for otherwise, by Theorem 5.1, $[L, L]$ would be bounded p -nil.

Now suppose that $p = 0$. Consider the exterior algebra E of an infinite dimensional F -space. Then E satisfies a PI: it is Lie nilpotent of class 2. Therefore, $R = E \otimes_F E \otimes_F E$ also satisfies a PI by a theorem of Regev [14]. Moreover, R is locally nilpotent since E is locally nilpotent. It was shown in [15], however, that R satisfies no nonmatrix identity and so, by Theorem 1.3, R° cannot satisfy a group identity. ■

Notice that by Kaplansky's theorem any nil example to this effect cannot be finitely generated.

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